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PROGRESS REPORT.

The following gentlemen have been elected members of the Society:

(1) Mr. C. P. Subrahmania Iyer, B.A., L.T.,—Teacher, Sarkar High School, Tripunithura (Cochin State)—(*at concessional rate.*)

(2) Mr. T. S. Sankara Iyer, B.A., B.E.,—Assistant Accountant-General, 50-51, Pycrofts' Road, Triplicane, Madras.

(3) Mr. W. A. Talbot, F.L.S.,—Conservator of Forests (retired), Springfield, Coonoor.

(4) Mr. C. Subrahmania Aiyar, B. A.,—Deputy Examiner of Accounts, North Western Railway, Lahore.

(5) Mr. T. P. Trivedi, M.A., LL.B.,—Professor of Mathematics, St. John's College, Agra.

(6) Mr. S. M. Shah, B.A.,—Fellow and M. A. Student, Gujarat College, Ahmedabad; (*at concessional rate.*)

(7) Mr. H. M. Dalal, B. A.,—M. A. Student, Gujarat College, Ahmedabad; (*at concessional rate*).

2. The Committee are highly gratified to find that the Indian Mathematical Society has been so well represented in the Coronation Honours' List.

Our Patron H. E. Sir George Clarke, Governor of Bombay is honoured with a *G.C.S.I.*, H. H. the Maharaja of Bhavnagar gets a *G.C.I.E.*, our retiring President Prof. B. Hanumanta Rao, a *Companionship of the Imperial Service Order*; Prof. F. M. Dastur of Bombay, a *Khan Bahadurship*; Prof. B. G. Naik of Karachi, a *Rao Sahebship*; and Mr. A. C. Pranatharthihara Aiyar, a *Rao Bahadurship.*

3. According to art. VIII (d) of our constitution the committee have appointed Messrs. M. T. Naraniengar M. A., and D. D. Kapadia M.A., B.Sc., as joint honorary secretaries for the year 1912.

4. The following new books have been purchased for the Library :

(1) *The Elements of the Theory of Algebraic Numbers*—by Prof. W. Reid, 15*s*. Published by MacMillan Co., New York.

(2) *Factor Tables for the First Ten Millions*—by Mr. D. N. Lehmer, £4/4*s*.—Published by Carnegie Institution of Washington.

(3) *Scientific Papers Vol. IV.—Periodic, Orbits and Miscellaneous Papers*—by Sir G. H. Darwin, 15*s*. Published by Cambridge University Press.

5. The following books have been presented to our Library by the Publishers, Messrs. A. Herman and Fils of Paris :

(1) *L. Equation de Fredholm* et ses *Applictions* a la *Physique Mathematique*—by Professor H. B. Heywood and M. Frechet.

(2) *Introduction a la Theori des Equations Integrales*—by Prof. T. Lalesco.

POONA, 1st Feb. 1912.

D. D. KAPADIA,
Hony. Joint Secretary.

The Incidence of an uncompensated Import Duty.

By B. J. Padsha, Bombay.

Problem.

To discuss the incidence of an import duty on corn, uncompensated by any duty on home-grown corn.

Data.

(*a*) Let there be n importing countries, and one exporting country; no cost of carriage within each country, but between the frontiers of the different countries a definite but arbitrary cost of carriage at any given time; let the importing countries levy each arbitrary duties on imported corn; the duty imposes on the importer further surcharges owing to capital and goods being locked up, and the harassment generally at the Customs Houses. The surcharge depends on the duty, but the mode of dependence in different countries is generally not the same. The whole cost of import is the duty, plus the cost of carriage, plus the surcharge.

[If d_r be the duty per unit of corn in the rth importing country, k_r the cost of carriage to it from the exporting country, k'_r the surcharge arising from the duty, and D_r the whole cost of transport we have

$$D_r = d_r + k_r + k'_r \qquad (a)$$

(altogether n equations)]

(*b*) The consumption of corn in each country is arbitrary at any time, but definite, and does not depend on price. The consumption in the importing countries is made up of the home-growth and the import; the production of the exporting country is equal to its own consumption plus what is exported abroad. Reserve stocks are neglected.

[If x_r be the home-growth in the rth importing country and X_r what it imports, and q_r its total consumption, we have

$$q_r = x_r + X_r \qquad (b)$$

(altogether n equations)

If $X = X_1 + X_2 + \dots\dots\dots + X_n$, $\qquad (b')$

and if q be the consumption of the exporting country, and x its total produce, we have $x = q + X \qquad (b')$

(*c*) Corn is taken to be of uniform quality; the growth in each country may be conceived to be split up into a large number of small and equal divisions. The different parts are raised at different costs; and it is easy to imagine an arrangement of the produce such that the first and last portions are the cheapest and dearest in cost, and the intermediate

show an ascending order of cost. The cost of production of any part then, depends on its place in this arrangement. The mode of variation in the arrangement will be different for different countries, and therefore arbitrary ; but is definite for each country at a given period.

[If c_r be the cost of production of the most costly part of the product in the rth country, we have

$$c_r = f_r(x_r) \qquad (c)$$

(altogether n equations).

We have also $\frac{d\,f_r(x_r)}{dx_r}$ positive

$$\text{or } f_r'(x_r) > o \qquad (c')$$

Again, we have

$$C = F(x) = F(q+X) \qquad (c'')$$

where C is the cost of production of the most costly portion in the exporting country. Cost includes the normal expected remuneration of labour and capital.

It is held to be impossible that at any given time, people will know better lands, and yet cultivate worse; therefore, in any given state of knowledge, a newer demand will in every country be met by raising *more* expensive products ; and any reduction of demand will be met by throwing out of supply the most costly portion. This statement of the Law of Diminishing Returns appears unobjectionabl.e]

(*d*) The selling price in each country must be equal to the cost of the most costly portion. It cannot be more; since the slightest excess would make profitable the resort to less fertile lands, increase supply, and prices will fall. It cannot be less, since in that case there would be no obligation to raise the most costly product.

[If p_r be the price per unit in the rth importing country, we have

$$p_r = c_r \qquad (d)$$

(altogether n equations),

and P = C (for the exporting country,) (d')

(*e*) The selling price in an importing country must equal the selling price in the exporting country *plus* cost of import.

$$p_r - P = D_r] \qquad (e)$$

(altogether n equations).

Demonstration.

We have altogether $(5n+4)$ independent equations thus :—

(a)	... ...	n
(b)	... ...	n
(b')	... ...	1
(b'')	... ...	1
(c)	... ...	n
(c'')	... ...	1
(d)	... ...	n
(d')	... ...	1
(e)	... ...	n
	Total	$5n+4$

These equations involve, at any given time a number of *known* quantities, the k's and k''s the d's, and the q's. k' is partly a function of d. The *forms* of the f's and F are also known. The unknown quantities are the n D's, $(n+1)$ x's, $(n+1)$ X's, $(n+1)$ c's, $(n+1)$ p's, altogether $(5n+4)$ independent unknown quantities. The values of all are therefore, determinate at any given time.

We proceed to determine the change in the unknowns due to a slight change (δd_1) in d_1. All the other given quantities are unaltered, except k_1' which in general becomes $k_1'+\delta k_1'$. We have therefore by differentiating (a),

$$\left.\begin{aligned} &\delta D_1=\delta d_1+\delta k_1' \quad \ldots \\ &dD_r=0 \ (r=2,\ldots n) \quad \ldots \end{aligned}\right\} \quad (\alpha)$$

From (e) and (α)

$$\left.\begin{aligned} &\delta p_1-\delta P=\delta D_1=\delta d_1+\delta k_1' \quad \ldots \\ \text{and } &\delta p_r-\delta P=\delta D_r=0 \ (r=2\ldots n) \quad \ldots \end{aligned}\right\} \quad (\varepsilon)$$

From (d) and (ε)

$$\left.\begin{aligned} &\delta c_r=\delta p_r=\delta P+\delta D_r=\delta C+\delta D_r \quad \ldots \\ \therefore\ &\delta c_1=\delta C+\delta D_1 \quad \ldots \\ \text{and } &\delta c_r=\delta C \ (\text{for } r=2\ldots n) \quad \ldots \end{aligned}\right\} \quad (\triangle)$$

Differentiating (c),

$$\delta c_r=f'_r(x_r)\delta x_r\ ; \quad \delta C=F'(q+X)dX\ ;$$

$\therefore$ by $(\triangle)$,

$$\left.\begin{aligned} &f_1'(x_1)\delta x_1=F'(q+X)\delta X+\delta D_1 \quad \ldots \\ \text{and } &f_r'(x_r)\delta x_r=F'(q+X)\delta X, \ (\text{for } r=2\ldots n) \quad \ldots \end{aligned}\right\}$$

As all the f's and F' are positive, the variations δX, $\delta x_2 \ldots\ldots \delta x_n$ are all of the same sign.

Differentiate (b);

$$\delta x_r = -\delta X_r \text{ for all}_r$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots$$

$$\text{and } \delta X = \delta X_1 + \delta X_2 + \ldots + \delta X_n$$

$$= -\delta x_1 - \delta x_2 - \ldots - \delta x_n \qquad (\beta)$$

$$\therefore \delta X + \delta x_2 + \ldots + \delta x_n = -\delta x_1$$

Now, δx_1 is the alteration in the home-growth arising from an uncompensated import duty; therefore it may possibly be zero, but is usually positive; and is never negative; $-\delta x_1$ is therefore never positive. Therefore the right hand side of equation (β) is either zero or negative. But by (γ) all terms on the left hand side are of the same sign; hence each term is either zero or negative.

$$\delta p_r = \delta P = \delta C = F' \, (q + X) \, \delta X.$$

Since F′ is positive, and δX is shown to be negative or zero, all the δp's (except δp_1) are negative or zero.

By (ε) $\delta p_1 = \delta D_1 + \delta P$ but ∂P is negative; therefore $\delta p_1 < \delta D_1$ or $\delta p_1 < \delta d_1 + \delta k_1'$.

That is to say, the price in the country which has altered its duty has been increased by an amount less than the alteration in the duty plus the surcharge; (and the prices in the exporting country and the countries where the duty has not been diminished have fallen, if, δx_1 is not zero).

If the duty is for revenue only, the production in the country is not affected; $\therefore \delta x_1 = 0$; hence every $\delta x = 0$, and $\delta X = 0$, and every δc, and δC are zero, except δc_1 which $= \delta d_1 + \delta k_1$.′ The only country in which the price is altered is therefore the country altering the duty, and the alteration of price is greater than the extra duty by an amount equal to the resulting extra surcharge.

The argument can be extended to δd_1 not small, *provided* $f'(x)$ remains positive throughout the setting down of prices.

II

The result of an alteration of duty in one importing country has been seen to be in general an alteration of prices in the whole corn-trading world. This involves transfer of a part of production to less favourable circumstances; but it also entails a redistribution of wealth as between country and country, and as between the producer, consumer and the treasury in each country. If we ask whether the country (1) which alters its import duty by $\delta . d$, will, on the whole, gain or lose, we must in reply set forth the following considerations. We assume that a gain of £ 1 to the consumer and the loss of it to the Treasury or

landlord can be set off against each other to arrive at the net gain or loss of the country.

The balance-sheet will read as follows:

Duty $= d_1$	Change when Duty $= d_1 + \delta d$.
Duty $= d_1$	$\delta.\ d_1$
Total receipts of the Treasury $= X_1\ d_1$	$\delta.\ (X_1\ d_1)$
Home growth : x_1	δx_1
Import :	$\delta\ X_1$
Consumer's payment : $(x_1 + X_1)\ p_1 = q_1\ c_1$	$q_1 \cdot \delta p_1 = q_1 \cdot \delta c_1$
Rent : $x_1 c_1 - \int_0^{c_1} c \delta x$	$x_1 . \delta c_1$
Gain	Loss
Rent . $x_1 . \delta c_1$	Consumer's payment :
Treasury : $\delta.\ (X_1 d_1)$	$(x_1 + X_1).\ \delta c_1$

$$\begin{aligned} \text{Net gain} &= \delta.\ (X_1 d_1) - X_1 \delta c_1 \\ &= X_1\ (\delta d_1 - \delta p_1) + d_1 . \delta X_1 \\ &= X_1 .\ (\delta d_1 - \delta p_1) - d_1 .\ \delta x_1 . \end{aligned}$$

When d_1 is zero, if the rise of price be less than the new small duty, then a small duty on imported corn means net gain to the country levying the duty; and this will always be the case if the surcharge be negligible, and the duty be protective to any extent; (for a protective duty involves an increased production, *i e.*, a positive $\delta\ x_1$). If the duty be one of revenue, then there is a loss equal to the surcharge.

III.

It may be argued that if a foreign duty on corn reduce English price on corn, and an English duty on corn reduce foreign price, then if all nations put duties on corn, it might be possible to reduce price of corn to the level of what it would have been under Universal Free Trade and still the duty might be retained. Could this possibility exist? Two remarks shoud be made. Free Trade England has lower prices when the rest of the world is not Free Trade,—lower prices than she would have if all nations put duties on corn, including England. Secondly, there is not only the reduction of prices, but also reduction of the protective effect of the duty, and reduction of rents. Prices cannot possibly be reduced without these accompaniments. Prices in each country *must* conform to the cost on the margin of cultivation. But if produc-

tion in the importing countries be reduced to the Universal Free Trade level, production in the exporting country would have to be brought up to that level also. And, then, there is a paradox. The price in the exporting country must be the price under Universal Free Trade; price in the importing country cannot be the same if there be an import duty. The truth is that each successive importing country's putting on a duty implies *less* reduction of prices in the exporting country; and the total reduction, therefore, never equal the duty. If it did, the revenues of all countries would profit without any other charge of any kind, and that is absurd. Or, to put the matter in another way. It is quite clear that if the size of the exporting area be fixed, then the reduction of price in the exporting area arising from duty in the importing area, increases with the size of the importing area; but not *proportionally* with the size The importing area *can never be* large enough to make the reduction of price in the exporting country equal to the duty. And what is not true with a single importing area cannot happen when the same area is subdivided into a resemble of national importing area.

IV.

It cannot be argued that because a small new duty on corn will in general result in a net gain to the hitherto freely importing country that therefore a large duty, making a succession of small doses of duty will do the same.

For the expression for net gain is proved to be positive only when the duty is small; in other cases it may be positive and negative according to the values of the other quantities involved.

V.

In the mathematical argument, $f'(x)$ is taken to be positive. Is it possible to extend the argument by assuming this function to be zero or negative? If it be, we should get a simple but complete general theory of taxation or imported communities, not connected by supply or demand with other commodities.

VI.

Opinions of publicists and politicians may now be treated.

VII.

We proceed to evaluate δP and δp_r.

$f_r(x_r+\delta x_r) = f_r(x_r)+f_r'(x_r)\delta x_r$, if δx_r is small.

and $= f_r(x_r)+f_r'(x_r+\Theta\,\delta x_r)\delta x_r$ if δx_r is not small,

where Θ is a positive proper fraction.

In either case, put

$$f_r(x_r+\delta\, x_r)=f_r(x_r)+\lambda_r\delta\, x_r.$$

The equations connecting the variations (small or large) are

$$\left.\begin{array}{l}\lambda_1\cdot\delta\, x_1=\delta\, P+\delta\, D_1\\ \lambda_r\cdot\delta\, x_r=\delta\, P\ (r=2 \quad n.)\\ L\,\delta\, x=\delta\, P.\end{array}\right\} \quad \cdots \quad \cdots \quad (A)$$

Also $\delta\, x_r+\delta\, X=o\ (r=1, 2\cdots n)$;

$$\therefore\quad \delta\, x=\delta\, X=-\sum_1^n\delta\, x_r \quad \cdots \quad \cdots \quad \cdots \quad \cdots \quad (B)$$

(A) and (E) give $n+2$ equation for $n+2$ unknowns

$$\delta\, x_1,\ \delta\, x_2, \ldots\ldots \delta\, x_n,\ \delta\, x,\ \delta\, P.$$

The solution is $\delta\, P/L=\delta\, x=-\sum_1^n\delta\, x_r=-\delta\, D_1/\lambda_1-\delta\, P\sum_1^n(1/\lambda_r)$

$$\therefore\ \cdot\ \delta\, P=-\delta\, D_1/\lambda_1\left[1/L+\sum_1^n(1/\lambda_r)\right]$$

This shows that δ P is negative, and numerically less than $\delta\, D_1$, as the denominator is greater than unity.

If the duty is small, $\delta\, x_r$ is small, and $\lambda_r=f_r'$. The above result is written

$$f_1'\frac{\partial P}{\partial D_1}=-1\Big/\left[1\Big/F'+\sum_1^r 1\Big/f_r'\right]=1\Big/M,\ \text{say.}$$

VIII.

We next consider the effect of a duty imposed in each importing country. First let the extra duty plus the surcharge in the rth country be small and equal to δD_r. As the variations above the first order are to be neglected,

$$\delta P=\sum^n\frac{\partial P}{\partial D_r}\delta D_r=-\frac{1}{M}\sum_1^n\frac{\delta D_r}{f'_r}$$

If all the δD_r are equal to say δD, we have

$$\delta P=-\delta D.\frac{1}{M}\left(\sum_1^n\frac{1}{f'_r}\right)=-\delta D.\frac{1}{M}\left(M-\frac{1}{F'}\right)$$

$$=-\delta D\left(1-\frac{1}{MF'}\right)=-\delta D\left(1-\frac{1}{1+F'\sum\frac{1}{f'_r}}\right)$$

where Q is a proper positive fraction, *i.e.*, the fall of prices in the exporting country never equals the new small duty plus surcharge, however large the importing area.

This result remains true when the change (say Δ) in the duty is not small; for

$$\text{the total fall of prices} = -\int \delta P = +\int_0^{\Delta} Q\delta D < \int_0^{\Delta} \delta D$$

$$< \int_0^{\Delta} \delta D, \text{ as } o < Q < 1$$

$$< \Delta$$

IX.

We shall now consider demand as generally elastic; thus

$$\begin{array}{l|l} X_1 + x_1 = q_1 = \phi_1(p_1) & \partial x_1 + \partial X_1 = \phi_1'(p_1)\partial p_1 \\ \cdots\cdots & \cdots\cdots \\ X_r + x_r = q_r = \phi_r(p_r) & \partial x_r + \partial X_r = \phi_r'(p_r)\partial p_r \\ \cdots\cdots & \cdots\cdots \\ x - X = q = \phi(P) & \partial x - \partial X = \phi'(P)\partial_r \end{array}$$

$$\therefore \quad \delta x = -\Sigma\delta x_r + \phi'(P)\delta P + \Sigma\phi_r'(p_r).\delta p_r$$

$$= \quad -\delta P.\Sigma\frac{1}{f_r'(x_r)} - \frac{\delta D_1}{f_1'(x_1)} + \phi'(P).\delta P$$

$$+ \sum^{n}\phi'_r(p_r).\delta P + \phi_1'(p_1)[\delta P + \delta D_1].$$

$$= \delta D_1\left\{\phi_1'(p_1) - \frac{1}{f_1'(x_1)}\right\} + \delta P\left\{\sum\phi_1'(p_r) + \phi'(P) - \sum\frac{1}{f_r'(x_r)}\right\}$$

$$\therefore \ \delta P = F'(x)\delta x = F'(x)\{\delta D_1(-\cdots) + \delta P(\ldots)\}$$

$$\therefore \ \delta P\left\{1 - F'(x)\left[\phi'(P) + \Sigma\phi_r'(p_r) - \Sigma\frac{1}{f_r'(x)}\right]\right\}$$

$$= F'(x).\delta D_1\left\{\phi_1'(p_1) - \frac{1}{f_1'(x_1)}\right\}$$

$$\therefore \quad \delta P \quad \frac{F'(x).\partial D_1\left\{\phi_1'(p_1) - \frac{1}{f_1'(x_1)}\right\}}{1 - F'(x)[\ldots \quad \ldots \quad \ldots]}$$

$$= \frac{\phi D_1}{f_1'(x_1)}\cdot\left[\frac{f_1'(x_1).\phi_1'(p) - 1}{M - \phi'(P) - \Sigma\phi_r'(p_r)}\right]$$

Since the ϕ's are negative, ∂P is negative. The larger $\phi_1'(p_1)$ is, the *greater* is the fall in price; the larger the other ϕ's are, the smaller is the fall of ice in the exporting country.

A Model illustrating the passage of a Light Wave through Quartz. *

By Professor H. Newman Allen, B.S., P.R.D.

(*Received: October, 1911.*)

In considering the passage of a light wave through a crystalline medium it is necessary in general to consider it, from the mathematical point of view, as being split up into two waves travelling with different velocities. In ordinary uniaxial crystals these two waves are plane polarised, in media which produce rotation of the plane of polarisation they are circularly polarised, while in crystals like quartz, which rotate the plane of polarisation when the wave travels along the axis, it becomes necessary, when the direction is oblique to the axis, to resolve into two elliptical vibrations with a definite ratio between the axes of the ellipses for each direction of propagation. The major axes of these two ellipses are at right angles to one another, and the direction of motion is clockwise in one and counter clockwise in the other.

Having arrived at the mathematical solutions of the different problems it seems worth while to consider a little more in detail the actual physical happening in the crystal. A particle cannot have two distinct and separate displacements at the same time; but the two mathematical displacements obtained above will in every case compound into a resultant displacement. If the crystal is so cut, and the wave sent through it in such a way, as to avoid separation of the two component waves by double refraction, it is easy to construct models showing how a vibration gradually alters in form as it passes through.

An ordinary helical or "spiral" stair case, for instance, serves well to illustrate rotary polarisation.

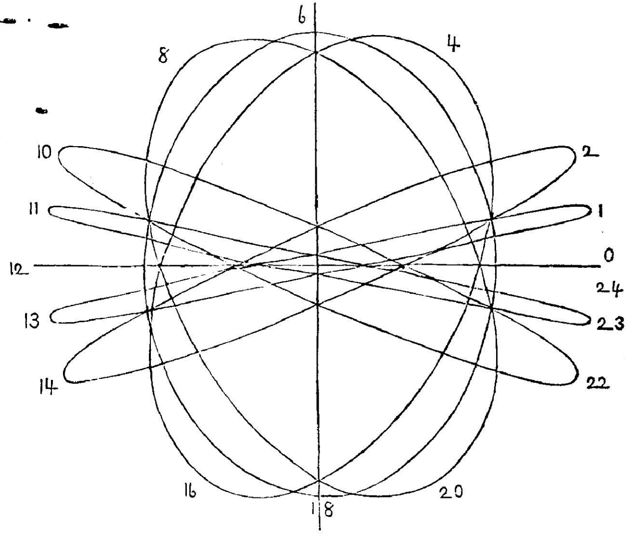
If a surface were constructed passing through the front edges of the stairs, it would be possible to draw upon it a series of instantaneous wave forms showing the propagation of a plane polarised wave through the medium.

Sections of a suitable model for ordinary double refraction by planes at right angles to the direction of propagation will show a series of ellipses with gradually varying ratios between their axes. Starting with a wave plane plarised at 45° to the principal plane we shall of course get the gradual opening up from elliptical into circular motion, the contraction into straight line motion inclined 90° to the original direction and so on, as we proceed along the axis of the model.

* This model was exhibited before the London Physical Society.

Here again lines of instantaneous wave form may be traced on the surface of the model.

One way of doing this, which answers easily well, is to cut out the ellipses and circles for a number of equidistant points along the line of propagation in cardboard or sheet metal, and mount them in position on a rod passing through the centre. By passing a series of coloured strings from ellipse to ellipse, through a series of suitable small holes at the edges, the required surface with the instantaneous wave forms can be shown. The ratio between the retardation and the wave length will have to be greatly exaggerated: but this is not a very important defect.



I have also constructed a model illustrating the passage of a wave originally plane polarised through a crystal which rotates the plane of polarisation of light sent along its axis (quartz). The direction of propagation is oblique to the axis of the crystal.

It is assumed that the crystal has been cut so that the surface of incidence is perpendicular to the direction of propagation of the wave.

The straight line 12—0 in the diagram indicates the direction of the original linear vibration. It gives also the direction of the major

axis of the counter clockwise rotating ellipse which can pass through without alteration of form. The numbered ellipses show the form of the vibration when the counterclock wise vibration has gained 1, 2, 4, 6, 8, 10, 11, 12 &c., twelfths of a period on the clockwise.

It should be mentioned that taking the double amplitude of the original vibration as five the major axis of the counter clockwise component ellipse is four units, the minor two, the major axis of the clockwise, which is at right angles to 12—0, is two units and the minor axis one.

It will be noticed that the ratio between the major and minor axes in each case has been assumed as two.

It will be seen from the diagram that, as the wave progresses, the linear vibration opens out into a counter clockwise elliptical one, that the major axis of this ellipse rotates in the counter clockwise direction, and that the minimum ratio of four to three between the axes is reached when the retardation is half a period. A retardation of a whole period gives a linear vibration again and so on. The model as constructed shows the effect of the first half period of retardation. The base of the model corresponds with curve 6 in the diagram, a horizontal plane two units above this cuts it in curve 4, another two units higher in curve 2, one unit higher gives curve 1, and still one unit higher brings us to the straight top edge of the model.

For the general case, where k, the ratio of the major to the minor axis of the invariable ellipses has values between 1 and ∞ depending on the inclination of the ray to the axis, it can be shown that the ratio between the axes of the ellipse marked 6 in the diagram is $\frac{2k}{k^2-1}$. This varies from infinity, when the light travels along the axis and the ellipse degenerates into a vertical straight line the plane of polarisation having been rotated 90,° to zero when k equals infinity and the ellipse becomes a horizontal straight line. This implies that a wave travelling at right angles to the axis and polarised in the principal plane or at right angles to it is transmitted unchanged.

For values of the k between 1 and $1+\sqrt{2}$ it appears that the major axis of the elliptical vibration into which the original linear vibration opens up is rotated constantly in the same direction as the light proceeds through the crystal. The latter value of k gives circularly polarised light with a retardation of half a period for the slower of the two component elliptical vibrations. For larger values of k the major axis oscillates on either side of the original direction of linear vibration

the amplitude of this oscillation becoming smaller and smaller as k increases.

It was hoped at first that the construction of this model might throw light on the action of the hypothetical revolving electrons in the crystal, whose orbits are supposed to be perpendicular to the axis. Qualitatively it will be seen that this hope has been at least partially realised.

At the same time it is clear that the propagation of certain elliptical vibrations unchanged through the crystal is a much simpler phenomenon than that described above and that a complete physical explanation of this in terms of the revolving electron theory is what is required.

It is not, however, by any means clear why, even if the retardation is produced by the passage of the waves obliquely through the circular orbits of the electrons, the major axes of the two invariable ellipses should be at right angles to one another, or why the ratio between the major and minor axes is the same for both of them.

Writing at a distance from a scientific library I am unable to find out whether the subject has been investigated mathematically in this direction. If not, such an investigation appears highly desirable.

REVIEWS:

Theoric des Series a termes constants. Applications aux calculs numeriques. E. Fabry. Published by A. Hermann et Fils, Paris, pp. 1—198; 6 fr. 50.

The usual discussion of the rules of convergency and divergency of series given in elementry treatises is generally far from satisfactory for two reasons: 1° it is generally assumed that there is a definite limiting value of the ratio $u_{n+1} : u_n$ as n tends to infinity; 2° the process of proceeding to the limit for the convergency tests is far from rigorous. As a simple illustration of the former point the series $\Sigma_n{}^{(-1)^n} x^n$ may be considered for which the ratios $u_{n+1} : u_n$ is alternately ∞ and o when n. is infinite, the series is never the less convergent for $o < x < 1$. The essential basis for testing the convergency or divergency of all series is comparison, and the comparison can in most simple cases be made without representing the tests as the evaluation of limiting values only.

In the book under review an elaborate discussion of the comparison tests is given in Chapter II. The fundamental comparison with the geometrical progression is given by the rules of D'Alembert and Cauchy; it is shown, in § 12, that the latter is more comprehensive

than the former. The rule of Raabe is next given with an extension that leads at once to a very simple and elegant proof of Gauss' celebrated test. (Gauss. Œuvres Completes III, p. 138). The name of Cauchy should be given as the discoverer of the test of § 19. The discussion of the famous Bertrand comparison series is given in §22 and proceeds similarly to that given in Jordan Cours D'Analyse. The use of the differential calculus for the proof of the inequalities required may be avoided by the formalist as being here unnecessary. The Bertrand series can be obtained in many other ways as is indicated later; but no student should fail to be acquainted with the well-known proof which is here given most clearly and rigorously. The Bertrand rules for convergency or divergency having been obtained, an example is given in § 28 in which they totally break down. Such examples have been constructed by du Bois-Reymond and Pringsheim. The example given may be enunciated as follows:

Defining e_r by the relations $e=e_1$, $e_2=e \ldots\ldots e_{r+1}=e^{e^r}$ and $l_r\, n$ by the relations $l_1 n=\log n$, $l_2\, n=\log l_1\, n$. $\ldots\ldots l_{+r1} n=\log l_r\ n$, choose a set of increasing integers v_1, $v_2 \ldots\ldots v_q \ldots\ldots$ where $v_q > e_q$, then the series

$$\sum_{1}^{v_1} \frac{1}{n^2} + \sum_{v_1+1}^{v_2} \frac{1}{n(l_1 n)^2} + \ldots\ldots + \sum_{v_q+1}^{v_{q+1}} \frac{1}{n\ l_1 n\ l_2 n \ldots l_{q-1}\ n(l_q\ n)^2} + \ldots$$

is such that the Bertrand rules give no indication regarding the convergency or or divergency of the series. The series is in fact convergent or divergent according as $\sum\limits_{q=1}^{\infty} \frac{1}{l_q(v_q)}$ is convergent or divergent. In particular if $e_q < v_q < e_{q+1}$, the series is divergent, but if, say $l_q\,(v_q) >$ 9 so that $v_1 > e_1$, $v_1 > e^{e^4}$, $v_3 > e^{e^{e^9}}$ ......, the series is convergent.

The well-known condensation test of Cauchy and similar tests due to Kummer and Ermahoo are briefly discussed and shown to include the Bertrand Series as special cases.

In Chapter III a very slight extension is made to include the discussion of semi-convergent series, the product of two infinite series and allied theorems. The author retains the name semi-convergent in the sense used by Legendre for those divergent series that can be used for the purpose of approximation. The more usual sense of the word is now so well established that it seems undesirable to make the proposed change.

In Chapter IV some interesting examples are given of summable series and the numerical evaluation and calculation of the remainder

of certain simple series are considered. It is well-known that slowly convergent series are useless for numerical evaluation and the problem of rendering such series more rapidly convergent by transformation is discussed in detail in Chapter V, where illustrations of the chief transformation methods due to Euler, Stirling, Kummer and Markoff are given. As an example of the power of these transformations the example treated in § 91 may be quoted. The series

$$1-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\ldots$$

is transformed by Kummer's method into

$$\frac{3919}{4410}+97\left(\frac{64}{5.7.9.11}\right)^2-52^{14}.6^4\sum_1^{\infty}\frac{(-1)^{n-1}}{(2n-1)^2(2n+1)^2\ldots(2n+11)^2}$$

giving the numerical approximation ·915, 965, 594...... a result that would require the calculation of more than 70,000 terms of the original series, whereas only 12 terms of the transformed series will do. Examples treated by Markoff's method carry the approximations to 24 places of decimals by the use of only a very small number of terms of the transformed series.

The concluding chapter deals with the use of divergent series for approximating to the numerical values of Euler's constant, Stirling's formula for log T (x), Euler's summation formula and certain definite integrals.

It will be seen from the above description of the general contents that M. Fabry's work contains in a small compass an account of many interesting and useful theorems, which are not to be found in any elementary English text book. The last two chapters are particularly valuable and throughout the book the utmost rigour is observed in regard to the proofs and limitations of the theorems enunciated. The first four chapters are quite elementary and will be found useful by many Indian students who possess the small knowledge of French necessary for reading a mathematical treatise.

A. C. L. W.

Projective Geometry for use in Colleges and Schools. William P. Milne, M.A., D. Sc. (1911, Macmillan & Co.)

This little book on Geometrical Projection is very suitable for class instruction. The private student, also, will find it as valuable an introduction to the methods of higher Pure Geometry as the same author's *Homogeneous Co-ordinates* is to the advanced theory of Analytical Conics.

The subject-matter is presented with much lucidity and commendable terseness; but the size of the book has precluded any sustained attempt at developing its study from the scholar's point of view.

The *circular points* at infinity have been fully treated and utilized. and the principle of *continuity* has been explained at some length. Though Homography and Involution have been considered to be outside the scope of the volume, almost all the main properties of Conics have been fairly accounted for. This the author has been enabled to do by introducing a convenient innovation. It is known that from Desargue's involution property of four-point conics may be deduced the theorem that two and only two conics can be drawn to pass through four points and to touch a given straight line. This proposition is here styled Desargue's, and preved by projection from the corresponding property of coaxal circles: it is then employed, along with its dual proposition, in the further development of the subject.

The exercises are abundant and well arranged. As was to be expected from the treatment adopted, they have been mostly selected from the College Scholarship Papers and the Tripos Examinations of the Cambridge University. The printing is excellent and the diagrams are neat and helpful.

K. J. Sanjana.

Elementary Mathematics, Part I: P. V. Seshu Aiyar, B.A., L.T., and V. Venkatasubbayya, B.A., L.T., (Srinivasavaradachari & Co., Madras, 1911).

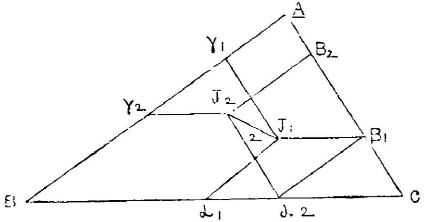
The introduction of the Secondary School-Leaving Certificate into this country appears to have set in operation forces likely to leave a permanent mark on the quality of Mathematical Teaching in Secondary Schools. The book under review is an attempt to remove the artificial distinction between Arithmetic, Algebra, and Geometry. The principles are carefully explained. The examples, which are numerous and of new types, are generally of a practical kind and relate to real problems of Indian life and conditions. The typography of the book is good for a local publishing firm and we gladly welcome this addition to the numerous modern text-books in Elementary Mathematics.

In a volume of nearly 500 pages the space devoted to Algebra is very meagre A reference Index seems desirable, as the work treats of diverse subjects.

SHORT NOTES:

The Jerabek Points.

To determine a point J_1, such that $J_1\alpha_1$, $J_1\beta_1$, $J_1\gamma_1$, drawn parallel to the sides of ABC, in the sense AB, BC, CA, may be equal.



Let $J_1\alpha_1 = x_1$:

and let $(p'q'r')$ be the normal $c's$ of J_1,

so that $ap' + bq' + cr' = 2\Delta$.

Then $a.x_1 \sin B + b.x_1 \sin C + c.x_1 \sin A = 2\Delta$.

$$\therefore \quad x_1 = \frac{4\Delta R}{bc+ca+ab} = \frac{abc}{bc+ca+ab}$$

$$\therefore \quad \frac{1}{x_1} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{2}{x}.$$

$$\therefore \quad x_1 = \tfrac{1}{2}x.$$

Also $p^1 = x_1 \sin B \propto b$.

$\therefore$ The normal $c's$ of J, are (b, c, a) and the bary-centric $c's$ are (ab, bc, ac) or $(1/c, 1/a, 1/b)$.

Similarly a point J_2 may be found, such that $J_2\alpha_2 = J_2\beta_2 = J_2\gamma_2$, the lines being drawn in the contrary sense AC, CB, BA.

Here $$\frac{1}{x} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

$$\therefore \quad x_1 = x_2 = \tfrac{1}{2}x:$$

$\therefore$ J_2 has normal $c's$ (c, a, b) and bar. $c's$ $(1/b, 1/c, 1/a)$.

J_1 and J_2 are called the Jerabek Points of ABC.

Since $\beta_1\gamma_1$ subtends at J_1, and $\alpha_2\gamma_2$ subtends at J_2, angles $\pi - C$ whose sides are x_1, x_2:

$$\beta_1\gamma_1 = \gamma_2\alpha_2, \text{ so, } \gamma_1\alpha_1 = \alpha_2\beta_2, \ \alpha_1\beta_1 = \beta_2\gamma_2.$$

Since $J_1\gamma_1$ is equals and parallel to $J_2\alpha_2$

$J_1\alpha_2$ is equal and parallel to $J_2\gamma_1$

Then from the triangles $J_1\alpha_1\alpha_2$, $J_2\gamma_1\gamma_2$, which have one side equal, and the other pairs of sides parallel

$$\alpha_1\alpha_2 = J_2\gamma_2 = x_1 = J_1\beta_1$$

∴ $J_1\alpha_1\alpha_2\beta_1$ is a rhombus:

∴ $J_1\alpha_2$ is perpendicular to $\alpha_1\beta_1$:

∴ $\gamma_2 J_2$ which is parallel to $J_1\alpha_2$, is perpendicular to $\alpha_1\beta_1$:

So $\alpha_1 J_2$ is perp^r to $\beta_1\gamma_1$.

∴ J_2 is the orthocentre of $\alpha_1\beta_1\gamma_1$: and it is the circumcentre of $\alpha_2\beta_2\gamma_2$

So J_1 is the circumcentre of $\alpha_1\beta_1\gamma_2$, and the orthocentre of $\alpha_2\beta_2\gamma_2$.

The point L, midway between the circumcentre and orthocentre, is the N. P. centre of both triangles. And since the triangles have equal circum-radii, they have the same N. P. circle.

23rd May 1911. W. GALLATLY.

Double Points.

Let ABC and A′B′C′ be two triangles, and P and P′ be the mean centres of A,B,C and A′, B′, C′ respectively for any common system of multiples. The points P and P′ may be said to be corresponding points of the triangles and the point which is its own correspondent may be called the double point of the triangles.

It is clear that by orthogonal projection, corresponding points and the double point project into corresponding points and the double point of the projections of the triangles.

2. To find the double point of ABC and A′B′C′:

Let P be the mean centre of A,B,C for the multiples l,m,n. Through P draw PL, PM, PN equal and parallel to AA′, BB′, CC′. The mean centres of L,M,N and A′, B′, C′ for the multiples l, m, n coincide, (*vide.* M'clelland's Geometry, page 94 Ex. 8). Now P is the mean centre of L,M,N for multiples proportional to the areas of the triangles MPN, NPL, LPM. Therefore when P is the double point, l,m,n are proportional to PM.PN sin MPN, PN.PL. sin NPL, PL.PM. sin LPM. Denoting AA′, BB′, CC′ by r_1, r_2, r_3 and the angles between BB′, CC′; CC′, AA′; AA′, BB′ by λ, μ, γ. The double point is the mean centre for the multiples $\frac{\sin\lambda}{r_1}$, $\frac{\sin\mu}{r_2}$, $\frac{\sin\gamma}{r_3}$. Denote the double point by D.

(i) The double point of a triangle ABC and its first Brocard triangle A′B′C′:—Consider the triangles ABC and C′A′B′. AC′ = $\frac{1}{2}$ c tan w and the angle between BA′ and CB′ = BΩC = A + B

Therefore $\frac{\sin\lambda}{r_1}=\frac{\sin(A+B)}{\frac{1}{2}c\tan w}=\frac{1}{R\tan w}$. The multiples are all equal and hence the double point of ABC and C′A′B is the double point of ABC and A′B′C′, *viz.*, their common median point.

(ii) The double point of a triangle ABC and its orthocentric triangle XYZ:—$Ax=\frac{2\Delta}{a}$ and $\lambda=180^\circ-A.\therefore\frac{\sin\lambda}{r_1}$, etc., are proportional to a^2, b^2, c^2. ∴ The double point is the symmedian point of ABC.

For the triangles ABC, YZX; ABC, ZXY. The multiples are $\frac{\sin B}{c\cos A}$, $\frac{\sin C}{a\cos B}$, $\frac{\sin A}{b\cos C}$; $\frac{\sin C}{b\cos A}$, $\frac{\sin A}{c\cos B}$, $\frac{\sin B}{a\cos C}$. The normal co-ordinates of the double points are $b^2\sec A$, $c^2\sec B$, $a^2\sec C$; $c^2\sec A$ $a^2\sec B$, $b^2\sec C$.

(iii) The double point of a triangle ABC and a triangle XYZ inscribed in and in perspective with ABC is the centre of the conic XYZ inscribed in ABC.

The symmedian point is the centre of the inconic circumscribing the orthocentric triangle, etc., project (ii).

3. If A″, B″, C″ be the verticies of directly similar triangles described on AA′, BB′, CC′, The mean centres of A″, B″, C″ for the multiples $\frac{\sin\lambda}{r_1}$, $\frac{\sin\mu}{r_2}$, $\frac{\sin\gamma}{r_3}$ is the double point D.

Since corresponding lines of directly similar figures intersect at a constant angle. The angles between the lines AA″, BB″, CC″ are λ, μ, γ. Also AA″ : BB″ : CC″ = AA′ : BB′ : CC′ = $r_1 : r_2 : r_3$. Therefore D is the common double point of the triangles ABC, A′B′C′, A″B″C″.

4. If three corresponding points of directly similar figures described on AA′, BB′, CC′ are collinear, the line of collinearity passes through D.

The mean centre of collinear points for any system of multiples lies on the line of collinearity. Their mean centre for the multiples $\frac{\sin\lambda}{r_1}$, etc., is fixed, being D. Hence all such lines pass through D.

Hence it follows that the double point of two triangles is the director point of similar figures described on the joins of corresponding vertices.

(i) The director point of similar figures described on the sides AB, BC, CA of a triangle ABC, is the double point of the triangles ABC, BCA, which is evidently the median point of the triangle.

(ii) The director point of similar figures on AX, BY, CZ where XYZ is the ortho-centric triangle of ABC is the double point of ABC and XYZ which is the symmedian point of ABC.

(iii) The double point of any triangle formed by corresponding points of three similar figures, and the invariable triangle is the director point of the figures; for the invariable points are corresponding points.

(iv) The orthopole of a line with respect to a triangle ABC is the double point of the pedal triangles of any two points P and P′ on the line.

Let LMN and L′M′N′ be the pedal triangles of P and P′ and L″M″N″, the pedal triangle of P″ on PP′. Since $\frac{LL'}{LL''}=\frac{MM'}{MM''}=\frac{NN'}{NN''}=\frac{PP'}{PP''}$ the points L″, M″, N″ are corresponding points of similar figures on LL′, MM′, NN′. When P″ is a point of intersection of PP′ with the circum-circle of ABC, L″, M″, N″ are collinear and therefore the pedal line of P″ passes through the double point, which for a similar reason, lies on the pedal line of the other point of intersection of PP′ with the circumcircle ABC.

30th August 1911. M. Bhimasena Rao.

The six-fold Systems of similar Circumscribed and Inscribed Triangles of a given triangle.

1. Let P be a point whose angle coordinates referred to a given triangle ABC are (λ, μ, γ); and let D, E, F be the centres of the circles BPC, CPA, APB. Then, if A′B′C′ be any *circumscribed* triangle whose sides make θ with those of DEF, the triangles A′B′C′, DEF are similar, P being the centre of similitude and $2\cos\theta$ the ratio of similitude.

2. Let P′ be the isogonal conjugate of P, and D′E′F′ the corresponding triangle for P′. Then, we know, that DEF, D′E′F′ are inverse, with respect to the circle ABC (*vide*: my paper in the April Journal Vol. III); and P′ is the centre of similitude of a system of *inscribed* triangles, similar to the circumscribed system whose centre of similitude is P.

3. Now, considering the circumscribed system and taking all possible permutations of (λ, μ, γ) we get a set of six points P_r $(r=1, 2, \ldots\ldots 6)$, such that each is associated with a definite system of similar triangles.

4. Again, if Q be the *twinpoint* of P with respect to ABC (*vide*, Gallatly, *Modern Geometry*, § 47), then the antipedal triangles of P & Q

are inversely similar. Thus, we have a second set of six points Q_r ($r=1, 2, \ldots\ldots 6$) such that each is associated with an *inversely* similar set of triangles.

5. Hence, we have a sixfold system of similar circumscribed triangles, each with a pair of twin points (P, Q) for centres of similitude.

6. Let (α, β, γ) be the normal coordinates of P, then

$$\tan\lambda = \pm \begin{vmatrix} 1/\alpha, & -1/\beta, & 0 \\ 1/\alpha, & 0, & -1/\gamma \\ \sin A, & \sin B, & \sin C \end{vmatrix} \div \left(\frac{1}{\alpha^2} - \frac{\cos A}{\beta\gamma} + \frac{\cos\beta}{\gamma\alpha} + \frac{\cos C}{\alpha\beta}\right)$$

$$= \pm(\alpha \sin A + \beta \sin B + \gamma \sin C)/(\beta\gamma/\alpha + \beta \cos B + \gamma \cos C - \alpha \cos A)$$

Therefore the six pairs of points P_r, Q_r ($r=1, 2, \ldots\ldots 6$) lie on

$$\Sigma(\beta\gamma/\alpha) + \Sigma(\alpha \cos A) \quad \pm \quad \Sigma(\cot\lambda).\ \Sigma(\alpha \sin A) = 0;$$

which denotes two bicircular quartics, as may be easily verified.

7. The isogonal transformation of the quartic in the last article is

$$\Sigma(\alpha^2) + \Sigma(\beta'\gamma \cos A) \quad \pm \quad \Sigma(\cot\lambda).\ \Sigma(\beta\gamma \sin A) = 0$$

and therefore denotes two circles coaxal with the circumcircle and the Brocard circle. Further, these circles are seen to be inverses of each other with respect to the circumcircle.*

In other words, the six pairs of points P_r', Q_r' ($r=1, 2\ldots\ldots 6$), corresponding to similar inscribed triangles, lie six and six on two inverse circles

8. Given the position of any *one* of these points the remaining points are uniquely determined by a simple geometrical construction as follows:—

Let 1 be the given point, and X the equibrocardian circle corresponding to it; and let X cut the circle B 1 C in 2, C 2 A in 3, A 3 B in 4, so on. There will thus be determined *five* definite points and no more, the last point 6 being cyclic with 3. In other words, B 12 C, B 45 C, B 36 C, &c., are all cyclic and any one of the six points 1, 2, 3......6 uniquely determines the remaining five. Also, by taking the inverse of X and 1, the conjugate set of six points on the inverse circle X′ is similarly determined†.

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* These are in fact the loci of points whose pedal triangles are equibrocardian with one another. (*Vide*: Casey's Conics p. 300, Ex. 12, where, however, only *one circle* is mentioned). The centres of these circles are inverses with respect to the Brocard Circle.

† The proof of this construction easily follows by inversion.

9. **Examples:**

(1) Consider a system of triangles similar to the original inscribed in ABC: in this case,

$\lambda + A = \mu + B = \gamma + C = 2$ right angles, and the equibrocardian circle, of § 7, become the Brocard circle, and its inverse *viz.*, the Lemoine axis. The first set of points coincides with the circum centre, Brocard pointss and the vertices of the second Brocard triangle; and the conjugate set will be the inverses of these lying on the radical axis of the Brocard circle and the circum-circle.

(2) when the inscribed triangles are equilateral the six pairs of points coincide with the limiting points of the coaxal system, *viz.*, the points whose angle coordinates are $A \pm 60°$, $B \pm 60°$, $C \pm 60°$.

(3) when the inscribed triangles are isosceles, there will be only *three* distinct pairs of points, determined by circles through two of the vertices A, B, C touching X or X′.

10. The six pairs corresponding to the *inscribed* system are such that the *inverses* of ABC with respect to any one of them are also similar triangles.

10th September 1911. M. T. NARANIENGAR.

The Face of the Sky for February, March and April 1912.

Sidereal time at 8 p.m.

	February.			*March.*			*April.*		
Date.	H.	M.	S.	H.	M.	S.	H.	M.	S.
1 ...	4	41	47	6	36	8	8	38	22
8 ...	5	9	23	7	3	44	9	5	58
15 ...	5	36	59	7	31	20	9	33	34
22 ...	6	4	35	7	58	56	10	1	10
29 ...	6	32	11	8	26	32	1	28	46

From this table the constellations visible during the evenings can be ascertained by a reference to their position as given in star-charts.

Phases of the Moon.

	February.				*March.*				*April.*			
	Date.	H.	M.		Date.	H.	M.		Date.	H.	M.	
Full Moon ...	3	5	28	A.M.	2	4	12	P.M.	2	3	35	A.M.
Last Quarter ...	10	6	21	,,	11	1	26	A.M.	10	1	54	,,
New Moon ...	18	11	14	,,	19	3	39	,,	17	5	10	P.M.
First Quarter ...	25	0	57	,,	26	8	32	,,	24	2	17	,,

Sun.

Enters the sign aries at 21 March at 5·0 A.M.

Planets.

Mercury attainssuperior conjunction with Sun on March 27 after which it is an evening star. It attains its greatest elongation (E) 18°5′ on March 28 at 7-30 A.M. On April 5, it is stationary and on April 15 it is in inferior conjunction with sun when it becomes a morning star and on April 28 it is stationary.

It is in conjunction with the moon on Feb. 17 and on April 17, with Venus on April 28, and with Uranus on Feb. 7.

Venus is still a morning star. It is in conjunction with the Moon on Feb. 15 at 4-0 A.M., on March 16 and on April 15 and with Uranus on Feb. 25.

Mars is in conjunction with the Moon on Feb. 26 at 9-2 P.M., on March 25 at 2-0 P.M. and on April 22 at 8-52 A.M. It is in quadratuer to Sun on March 4.

Jupiter is a morning star. It is in conjunction with the Moon on Feb. 12 at 2-51 A.M., on March 10 at 4-6 A.M and April 7 at 2-2 A.M It is in quadrature to Sun on March 5. It is stationary on April 1.

Saturn is in quadrature to Sun on Feb. 4, in conjunction with the Moon on Feb. 24 at 1-58 P.M., on March 22 at 11-49 P.M. and April 19 at 0-1 P.M.

Uranus is in conjunction with the Moon on Feb. 16 and on April 10.

Neptune is in conjunction with the Moon on Feb. 1 at 7-6 P.M. and on March 27.

V. RAMASAM.

SOLUTIONS.

Question 86.

(M. T. NARANIENGAR, M. A.) :—*Corrected*: With the usual notion, prove that the nine-point circles of O, K, I (or I_1, I_2, I_3) are concurrent with the circumcircle of ABC.

Solution by 'Zero'.

The rectangular hyperbola

$$(b^2-c^2)\alpha^2+(c^2-a^2)\beta^2+(a^2-b^2)\gamma^2=0,$$

Passes through the points O,K,I, I_1, I_2,I_3, and its centre is the Euler point $\{ a/(b^2-c^2), b/(c^2-a^2), c/(a^2-b^2) \}$, which is the common point of the nine point circles in question.

Question 248 (i).

(S. NARAYANAN, B. A., L. T.) :—Shew that the Feuerbach tangents also touch the maximum inscribed ellipse of the triangle.

[*Vide*: Solution of Q. 221 on *p*. 34 of Vol. III of the Journal].

Question 263.

263. (M. BHIMASENA RAO).—A variable triangle ABC is inscribed in a conic X and circumscribed about another conic Y. If the isogonal conjugate of X with respect to ABC in any one of its positions touches Y, the isogonal conjugate of X with respect to ABC in any other position also touches Y.

Solution by N. Sankara Aiyar, B.A.

Consider the circular points at infinity ω, ω'. All conics through them are circles.

Now, let S be the focus of any parabola.

The triangle $S\omega\omega'$ is circumscribed to the parabola. For, the tangents from S to the parabola are the isotropic lines through S and $\omega\omega'$ is the line at infinity. Hence, if the sides of any triangle touch the parabola, its three vertices and S, ω, ω' lie on a conic—which is therefore a circle,

[This furnishes an elegant proof of the theorem that if ABC be a triangle of tangents to a parabola whose focus is S, then SABC are concyclic].

The isogonal conjugate of a circle with respect to a triangle formed by three points on it is the line at infinity. Hence, if ABC lie on a circle though ABC, and the sides of ABC touch a parabola whose focus is S, the line at infinity touches it and this is the isogonal conjugate of the circle with respect to ABC. Hence by projection we get that if a conic X is circumscribed to Y and its isogonal conjugate with respect to one inscribed triangle touches Y, then its isogonal conjugate with respect to any inscribed triangle also touches it.

Question 302.

(D. D. KAPADIA, M. A., B. Sc.) :—On two edges AB, CD, of a skew quadrilateral AB, CD, points P, Q are taken so that AP : PB = DQ : QC. Prove by taking DA, DB, DC as axes of x, y, z that PQ is a generator of a hyperbolic paraboloid the equation of which can be put down in the form

$$\left(\frac{x}{a}+\frac{y}{b}\right)\left(\frac{y}{b}+\frac{z}{c}\right)=\frac{y}{b}.$$

Solution by N. Sankara Aiyar and K. S. Appaswami.

Let AP : PB $=\lambda=$ DQ : QC and let DA, DB, DC equal a, b, c respectively.

$$\therefore \quad \text{Q is } 0,\ 0,\ \frac{\lambda c}{\lambda+1}. \quad \text{and}$$

$$\text{P is } \frac{a}{\lambda+1},\ \frac{\lambda b}{\lambda+1},\ 0.$$

Let PQ be
$$z=px+q$$
$$=ry+s.$$

and we get by substitution of co-ordinates of P, Q, these equations of PQ to be

$$az+c\lambda x=\frac{a\lambda c}{\lambda+1}, \qquad bz+cy=\frac{b\lambda c}{\lambda+1}$$

$$i.\ e., \quad \frac{y}{b}+\frac{z}{c}=\frac{\lambda}{\lambda+1} \qquad \text{(i).}$$

$$\frac{\lambda x}{a}+\frac{z}{c}=\frac{\lambda}{\lambda+1}$$

$$\therefore \quad \lambda=\frac{ay}{bx}$$

$$\therefore \quad \frac{\lambda}{\lambda+1}=\frac{\frac{y}{b}}{\frac{y}{b}+\frac{x}{a}}.$$

Substituting in (i)

$$\left(\frac{y}{b}+\frac{z}{c}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{y}{b}.$$

The opposite system of generators are given by.—

$$\frac{y}{b}+\frac{x}{a}=\frac{\lambda+1}{\lambda}$$

$$\frac{y}{b}+\frac{z}{c}=\frac{y}{b}\quad\frac{\lambda}{\lambda+1}.$$

Hence these are got by joining P′Q′ when P′, Q′ are on DA, BC, so that

$$\frac{\mathrm{DP}'}{\mathrm{P}'\mathrm{A}}=\frac{\mathrm{BQ}'}{\mathrm{Q}'\mathrm{A}}=-\gamma-1$$

e.i., P′ and Q′ are on DA and CB produced.

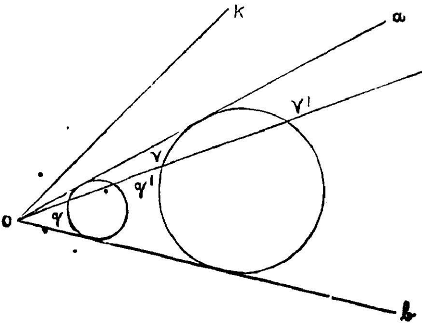
Question 303.

(T. Rajarama Rao, B.A., B.L.) :—Any two conics are drawn with a fixed point S as one of their foci, passing through two other fixed points A and B. P is a fixed point on BA produced. From P tangents PQ and PQ′ are drawn to one of the conics, and tangents PR, PR′ are drawn to the other conic. Find the locus of the intersection of QR′ and RQ.

Solution by N. B. Pendse and N. Sankara Aiyar.

Reciprocate with respect to S then the proposition will run thus.

There are three concurrent straight lines *a*, *b*, *p*. Any two circles are drawn touching *a* and *b*. They cut *p* in points Q,Q′, R,R′. Tangents at Q and R′ meet in L. Tangents at Q′ and R meet in M: Find the envelope of LM.



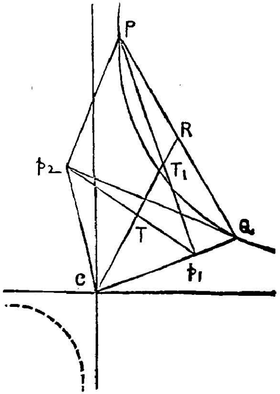
In this case. Find ok such that $o(pk, ab)$ are harmonic.

Tangents q, r are parallel to a fixed direction and tangent at $q'r'$ are parallel to another fixed direction. The poles of qq' and rr' are upon ok.

Therefore LM is the diagonal of a parallelogram whose second diagonal is ok and whose sides are parallel to fixed directions.

Therefore LM is parallel to a fixed direction. Therofore the envelope of LM is a fixed point on the line at infinity.

Reciprocating back: the locus of the meet of QR′ and Q′R in the conics is a straight line passing through S.



Question 306.

(M. T. NARANIENGAR, M.A.) :—P,Q are points on a rectangular hyperbola with centre c. If perpendiculars Pp_1, Pp_2 are drawn on CQ and the tangent at Q, prove geometrically $Cp_1 = Cp_2$; and generalise the result.

Solution by A. A. Krishnaswami Iyengar and N. B. Pendse.

The $\angle s$ at p_1 and p_2 are right angles. Therefore $Pp_2\, p_1Q$ is cyclic and the centre of the circle is the mid pt. of PQ (say R).

In a rect. hyp. the angle between two diameters is equal to the angle between their conjugates, so that

$$R\hat{C}Q = R\hat{Q}p_2 = P\hat{p_1}p_2$$

If CR meets $p_1 p_2$ in T

$$R\hat{T}p_1 = R\hat{C}p_2 + C\hat{p}_1T = P\hat{p}_1T + \hat{C}p_1T = \text{a right angle}.$$

Hence CR bisects p_1p_2 perpendicularly and $Cp_1 = Cp_2$.

Question 318.

(K. J. SANJANA, M.A.) :—Prove that :—

$$\int_0^{\pi/2} \log \sin \theta . \log \tan\theta \, d\theta = \frac{\pi^3}{16};$$

$$\int_0^{\pi/2} \log \sin \theta . \log \cos \theta \, d\theta = \frac{\pi}{2} \left\{ (\log 2)^2 - \frac{\pi^2}{24} \right\};$$

and
$$\int_0^{\pi/2} \theta \log \sin \theta d\theta = \tfrac{1}{2} \left\{ -\frac{\pi^2}{4} - \log 2 + \frac{1}{1^3} + \frac{1}{3^3} + \frac{1}{5^3} + \dots \right\}$$

[Suggested by Quest. 308 of Mr. S. RAMANUJAN ; the first two integrals are given by *Wolstenholme.*]

Solution by V. B. Naik M.A., R. Srinivasan, M.A., and J. C. Swaminarayan, M.A.

(318 *a*) $\log \sin \theta = -\log 2 - \sum_1^\infty \frac{1}{n} \cos 2n\theta$

$= -\log 2 - S$ (say).

put $\frac{\pi}{2} - \theta$ for θ, we get

$$\log \cos \theta = -\log 2 + \sum_1^\infty \frac{(-1)^{n-1}}{n} \cos 2n\theta$$

$= -\log 2 + S'$ (say)

$\therefore$ $\log \tan \theta = -(S+S')$

$\therefore$ $\log \sin \theta . \log \tan \theta = \log 2. (S+S') + S(S+S')$

$\therefore$ The effective part in the integral is $S(S+S')$

which $= (\cos 2\theta + \frac{1}{2} \cos 4\theta + \dots) 2(\cos 2\theta + \frac{1}{3} \cos 6\theta + \dots)$

and the effective part in this is $2 (\cos^2 2\theta + \frac{1}{9} \cos^2 6\theta + \frac{1}{25} \cos^2 10\theta + \dots)$

which $= (1 + \cos 4\theta) + \frac{1}{3^2} (1 + \cos 12\theta) + \dots$

and the effective part in this is $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} \dots = \frac{\pi^2}{8}$

$\therefore$ The integral $= \frac{\pi^2}{8} \cdot \frac{\pi}{2} = \frac{\pi^3}{16}.$

Solution by R. Srinivasan, M.A.

(318 *b*) $\log\sin\theta\log\cos\theta=(\log 2)^2+\log 2\,(S-S')-S\,S'$.

$\log 2\,(S-S')$ is ineffective, and the effective part in $S\,S'$ is

$$\cos^2 2\theta-\frac{1}{2^2}\cos^2 4\theta+\frac{1}{3^2}\cos^2 6\theta\quad-\quad\ldots$$

$$=\frac{1}{2}\left\{(1+\cos 4\theta)-\frac{1}{2^2}(1+\cos 8\theta)\quad+\quad\ldots\right\}$$

and the effective part in this is

$$\frac{1}{2}\left(\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}\quad-\quad\ldots\quad\ldots\right)$$

$$\frac{1}{2}\cdot\frac{\pi^2}{12}=\frac{\pi^2}{24}$$

$\therefore$ The required integral is

$$\frac{\pi}{2}\left\{(\log 2)^2-\frac{\pi^2}{24}\right\}.$$

(318 *c*) $\log\sin\theta=-\log 2-\sum\frac{\cos 2n\theta}{n}$

$$\therefore\int\theta\log\sin\theta\,d\theta=-\log 2\int\theta\,d\theta-\sum\frac{1}{n}\int\theta\cos 2n\theta\,d\theta$$

$$\text{Now}\int\theta\cos 2n\,\theta\,d\theta=\frac{\theta\sin 2n\,\theta}{2n}-\frac{1}{2n}\int\sin 2n\,\theta\,d\theta$$

$$=\frac{\theta\sin 2\,n\theta}{2n}+\frac{\cos 2\,n\theta}{4n^2}$$

$$\therefore\int_0^{\frac{\pi}{2}}\theta\log\sin\theta\,d\theta=-\log 2\left(\frac{\theta^2}{2}\right)_0^{\frac{\pi}{2}}$$

$$-\sum\frac{1}{4n^3}\left(\cos 2n\theta\right)_0^{\frac{\pi}{2}}\quad\text{(the rest vanishing.)}$$

$$=-\frac{\pi^2}{8}\log 2+\sum\frac{1}{4n^3}\{1-(-1)^n\}$$

$$=\tfrac{1}{2}\left\{-\frac{\pi^2}{4}\log 2+\frac{1}{1^3}+\frac{1}{3^3}+\frac{1}{5^3}+\ldots\right\}$$

Question 320.

(N. Sankara Aiyar) :—Shew that the tangents to a system of conics inscribed in a given quadrilateral at the points where a fixed straight line meets them envelope a curve of the third class touching the given line and the sides of the given quadrilateral. [Math. Tripos 1885.]

Solution by V. B. Naik M.A., and M. Bhimasena Rao.

Let the eqn. of the system of conics inscribed in the quadrilateral be referred to the harmonic triangle. It is

$$Ax^2+By^2+Cz^2=o$$

with the condition $\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=o$, so that the sides of the quadrilateral are $(\pm l,\ \pm m,\ \pm n)$.

Let $Lx+My+Nz=o$, be the fixed line which cuts one of the conics, say in the point, $(\xi,\ \eta,\ \zeta)$. The tangent at $\xi,\ \eta,\ \zeta$ is $A\xi x+B\eta y+C\zeta x \equiv \lambda x+\eta y+\nu z=o$.

So that $\frac{A\xi}{\lambda}=\frac{B\eta}{\mu}=\frac{C\zeta}{\nu}$. Hence since the point (ξ,η,ζ) lies on the conic and the line $Lx+My+Nz=o$, we have

$$\frac{\lambda^2}{A}+\frac{\mu^2}{B}+\frac{\nu^2}{C}=o$$

and
$$\frac{L\lambda}{A}+\frac{M\mu}{B}+\frac{N\nu}{C}=o$$

We also have
$$\frac{l^2}{A}+\frac{m^2}{B}+\frac{n^2}{C}=o$$

$$\therefore \quad \begin{vmatrix} \lambda^2 & \mu^2 & \nu^2 \\ L\lambda & M\mu & N\nu \\ l^2 & m^2 & n^2 \end{vmatrix}=o$$

which is the tangential equation of the envelope of $(\lambda\mu\nu)$. It is of the 3rd class, and it touches the sides of the quadrilateral, since by putting $\lambda=\pm l$, $\mu=\pm m$, $\nu=\pm n$, the determinant vanishes.

Question 322.

(G. RAMACHANDRAN) :—AXYB is a semicircle described on AB as diameter. From X and Y perpendiculars XP and YQ are drawn to AB. A circle is drawn touching the semicircle externally and PX and QY produced. If OM be the perpendicular from O, the centre of the circle, on AB, shew that $OM^2=AQ.BP$.

Solution by A. A. Krishnaswami Iyengar, K. D. Shah and others.

The radius of the touching circle is evidently $=PM=QM=\frac{1}{2}PQ$.

$\therefore$ If C be the centre of the semicircle, $CO=CA+PM$,

and
$$\begin{aligned} OM^2 &= CO^2-CM^2 \\ &=(CO+CM)(CO-CM) \\ &=(CA+PM+CM)(CA+PM-CM) \\ &=(BC+CM+MP)(AC-CM+MQ) \\ &= BP.\ AQ. \end{aligned}$$

Question 326.

(N. SANKARA AIYAR.) :—Prove that

$$\frac{|n}{|n-p+1} - p.\frac{|n-m}{|n-p-m+1} + \frac{p.\,p-1}{1.2}\frac{|n-2m}{|n-p-2m+1} - \dots = 0$$

Solution by V. B. Naick, and R. Srinivasan.

Multiplying the expression by $|p-1$, we see that we have to show that

$${}_nC_{p-1} - {}_pC_1 \cdot {}_{(n-m)}C_{p-1} + {}_pC_2 \cdot {}_{(n-m)}C_{p-1} - \quad = 0$$

Now this is the coefficient of x^{p-1} in

$$(1+x)^n - {}_pC_1(1+x)^{n-m} + {}_pC_2(1+x)^{n-2m} - \dots\dots$$

i. e. $(1+x)^n[1-{}_pC_1(1+x)^{-m}+{}_pC_2(1+x)^{-2m}-\dots\dots]$

i. e. $(1+x)^n\left[1-\frac{1}{(1+x)^m}\right]^p$, provided $n-pm \not< p-1$

i. e. $\frac{(1+x)^n[(1+x)^m-1]^p}{(1+x)^{mp}}$

i. e. $\frac{(1+x)^n[m^px^p+\dots\dots\dots]}{(1+x)^{mp}}$,

which shows that this coefficient $=0$

Question 330.

(K. J. SANJANA.) :—Prove that

(i) $1.\frac{1}{1}+\frac{1}{2}.\frac{1}{3}+\frac{1.3}{2.4}.\frac{1}{5}+\frac{1.3.5}{2.4.6}.\frac{1}{7}=\dots \quad =\frac{\pi}{2}$;

(ii) $1.\frac{1}{1^2}+\frac{1}{2}.\frac{1}{3^2}+\frac{1.3}{2.4}.\frac{1}{5^2}+\frac{1.3.5}{2.4.6}.\frac{1}{7^2}+\dots \quad =\frac{\pi}{2}\log 2$

and (iii) $1.\frac{1}{1^3}+\frac{1}{2}.\frac{1}{3^3}+\frac{1.3}{2.4}.\frac{1}{5^3}+\frac{1.3.5}{2.4.6}.\frac{1}{7^3}+\dots \quad =\frac{\pi}{4}\left\{(\log 2)^2+\frac{\pi^2}{12}\right\}$

[*Note* —It is not easy to evaluate similar series with higher powers I find.

$$1.\frac{1}{1^4}+\frac{1}{2}.\frac{1}{3^4}+\frac{1.3}{2.4}.\frac{1}{5^4}+\frac{1.3.5}{2.4.6}.\frac{1}{7^4}+\dots$$

$$=\frac{\pi}{12}\log 2\left\{(\log 2)^2+\frac{\pi^2}{4}\right\}+\frac{5}{48}\sum\frac{1}{r^3}+\frac{1}{4}\sum\frac{1}{rst}.$$

Here r has integral values from 1 to ∞; in the last term $r+s=t$ and $r+t\neq s$.]

Solution by V. B. Naik, M.A., J. C. Swaminarayan, M.A., and R. Srinivasan, M.A.

(i) follows by putting $x=1$ in the expansion for $\sin^{-1}x$.

(ii) $\frac{\sin^{-1}x}{x}=1+\frac{1}{2}.\frac{x^2}{3}+\frac{1.3}{2.4}.\frac{x^4}{5}+\dots$

Integrate both sides with respect to x between the limits 1 and o.

$$\text{Now} \int_0^1 \frac{\sin^{-1}x}{x} dx = \int_0^{\frac{\pi}{2}} \theta \cot\theta \, d\theta$$

$$= \Big[\theta \log \sin \theta\Big]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \log \sin \theta \, d\theta$$

$$= -\left(-\frac{\pi}{2} \log 2\right) = \frac{\pi}{2} \log 2$$

and the R.H.S, becomes equal to the L.H.S. of the question.

(iii) Again the L.H.S. here equals

$$\int_0^1 \frac{dx}{x} \int_0^x \frac{\sin^{-1}x}{x} dx$$

$$= \int_0^{\frac{\pi}{2}} d\theta \cot \theta \left\{ \theta \log \sin \theta - \int_0^{\theta} \log \sin \theta d\theta \right\}$$

$$= \int_0^{\frac{\pi}{2}} \theta \cot \theta \log \sin \theta d\theta$$

$$- \int_0^{\frac{\pi}{2}} d \theta \cot \theta \int \log \sin \theta \, d\theta$$

$$= \frac{1}{2} \left\{ \theta \log \sin \theta)^2 \right\}_0^{\frac{\pi}{2}} - \frac{1}{2} \int_0^{\frac{\pi}{2}} (\log \sin \theta)^3 d\theta$$

$$- \left\{ \log \sin \theta \int \log \sin \theta d\theta \right\}_0^{\frac{\pi}{2}}$$

$$+ \int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta$$

Here the first term vanishes ; and the third term equals zero also.

$$\therefore \quad \text{The L. H. S} = \frac{1}{2} \int_0^{\frac{\pi}{2}} (\log \sin \theta)^2 d\theta$$

$$= \frac{\pi}{4} \left\{ (\log 2)^2 + \frac{\pi^2}{12} \right\}$$

(See solution to question 308 by Mr. K. J. Sanjana)

Further remarks by V. B. Naick M. A.

$$\int_0^1 \frac{(\log x)^3}{\sqrt{(1-x^2)}}\,dx = \int_0^{\frac{\pi}{2}} (\log \sin \Theta)^3 d\Theta = -\frac{\pi}{2}\log 2 \left[(\log 2)^2+\frac{\pi^2}{4}\right] - \int_0^{\frac{\pi}{2}} Q^3 d\Theta,$$

Where $Q \equiv \cos 2\Theta + \frac{1}{2}\cos 4\Theta + \frac{1}{3}\cos 6\Theta + \ldots \left(\Theta < \Theta < \frac{\pi}{2}\right)$.

Expanding $(1-x^2)^{-\frac{1}{2}}$ and observing that $\int_0^1 x^m (\log x)^3 dx = -\frac{6}{(m+1)^4}$,

we have,

$$1.\ \frac{1}{1^4} + \frac{1}{2}\frac{1}{3^4} + \frac{1\cdot 3}{2\cdot 4}\ \frac{1}{5^4} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{1}{7^4} + \ldots\ldots\ldots$$

$$= \frac{\pi}{12}\log 2 \left\{(\log 2) + \frac{\pi^2}{4}\right\} + \frac{1}{6}\int_0^{\frac{\pi}{2}} Q^3 d\,\Theta.$$

$$\text{Now } Q^3 = \Sigma\frac{\cos^3 2r\Theta}{r^3} + 3\Sigma\frac{\cos^2 2r\Theta\cos 2s\Theta}{r^2 s} + 6\Sigma\frac{\cos 2r\Theta\cos 2s\Theta\cos 2t\Theta}{r\,s\,t}.$$

Where, r, s, t are integers in ascending order of magnitude. also,

$4\cos^3 2r\Theta = \cos 6r\Theta + 3\cos 2r\Theta$:

$2\cos^2 2r\Theta\cos 2s\Theta = \cos 2s\Theta + \cos 4r\Theta\cos 2s\Theta$.

$2\cos 2r\Theta\cos 2s\Theta = \cos 2(r+s)\Theta\cos 2t\Theta + \cos 2(s-r)\Theta\cos 2t\Theta$.

$= \cos 2(r+t)\Theta\cos 2s\Theta + \cos 2(t-r)\Theta\cos 2s\Theta$.

We observe that $\int_0^{\frac{\pi}{2}} \cos 2m\Theta\cos 2n\Theta = 0$, if $m \neq n$.

$\frac{\pi}{4}$ if $m = n$.

Then $\int_0^{\frac{\pi}{2}} \cos^3 2r\Theta\, d\Theta = 0$.

$$(1)\int_0^{\frac{\pi}{2}} \cos 2r\theta \cos s\theta\, d\theta = o \text{ if } s \neq 2r.$$

$$= \frac{\pi}{4} \text{ if } s = 2r.$$

$$(2)\int_0^{\frac{\pi}{2}} \cos 2r\theta \cos 2s\theta \cos 2t\theta\, d\theta = o, \text{ if } r+s \neq t.$$

$$= \frac{\pi}{4}, \text{ if } r+s=t.$$

N.B.—$r+t$ cannot be equal to s, nor $s-r$ to t, since, r, s, t are supposed to be written in ascending order of magnitude.

$$\therefore \int_0^{\frac{\pi}{2}} Q^3 d\theta = \frac{3}{2} \times \frac{\pi}{4} \Sigma \frac{1}{r^2 s} + 3, \frac{\pi}{4} \Sigma \frac{1}{r\, st}$$

where in the first summation $s = 2r$, and in the second, $r+s=t$,

$$= \frac{3\pi}{16} \Sigma \frac{r^3}{1} + \frac{3\pi}{4} \Sigma \frac{1}{r\, st}$$

where in the first summation r has integral values from 1 to ∞, and in the second $r+s=t$. In the latter we may separate those cases in which $r+s=t$ and at the same time $r+t=2s$, so that $2r=s$ and $3r=t$. Then the integral becomes.

$$= \frac{3\pi}{16} \Sigma \frac{1}{r^3} + \frac{3\pi}{4} \cdot \frac{1}{1 \cdot 2 \cdot 3} \Sigma \frac{1}{r^3} + \frac{3\pi}{4} \Sigma \frac{1}{r\, s\, t}$$

where in the last term $r+s=t$ and $r+t \neq 2s$.

This $= \frac{5\pi}{16} \Sigma \frac{1}{r^3} + \frac{3\pi}{4} \Sigma \frac{1}{r\, s\, t}$, with the above conditions.

$\therefore$ The series given $=$

$$\frac{\pi}{12} \log 2 \left\{ (\log 2)^2 + \frac{\pi^2}{4} \right\} + \frac{\pi}{2} \left\{ \frac{5}{48} \Sigma \frac{1}{r^3} + \frac{1}{4} \Sigma \frac{1}{r\, s\, t} \right\}$$

It appears that the factor $\frac{\pi}{2}$, should be inserted in the latter portion of the answer given in Prof. Sanjana's question.

Question 336.

(K. J. Sanjána, M.A.) :—Shew that

$$(i)\int_0^{\pi/2} (\log \sin\theta)^r d\theta = (-1)^r \underline{|r} \left\{ \frac{1}{1^{r+1}} + \frac{1}{2} \cdot \frac{1}{3^{r+1}} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{1}{5^{r+1}} + \cdots \right\};$$

(ii) $\int_0^{\pi/2} (\log \sin\theta)^r \sec\theta d\theta = (-1)^r \lfloor r \left\{ \frac{1}{1^{r+1}} + \frac{1}{3^{r+1}} + \frac{1}{5^{r+1}} + \ldots \right\}$;

(iii) $\int_0^{\pi/2} (\log \sin\theta)^r \frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta} d\theta = (-1)^r \lfloor r \left\{ \frac{1}{1^{r+1}} + \frac{1}{2^{r+1}} + \frac{1}{3^{r+1}} + \ldots \right\}$.

Also evaluate the integrals when r is odd. [$(-1)^r$ on the right will disappear on putting log cosec θ on the left side.]

Solution by S. Ramanujan and R. Srinivasan, M.A.

(i) $\int_0^{\frac{\pi}{2}} (\log \sin \theta)^r \, d\theta = \int_0^1 \frac{(\log x)^r}{\sqrt{1-x^2}} dx$

changing x to e^{-y} the integral becomes

$$\int_0^\infty \frac{(-y)^r}{\sqrt{e^{2y}-1}} dy = (-1)^r \int_0^\infty y^r (e^{-y} + \frac{1}{2} e^{-3y} + \frac{1\cdot 3}{2\cdot 4} e^{-5y} + \&c.) \, dy.$$

$$= (-1)^r \lfloor r \left(\frac{1}{1^{r+1}} + \frac{1}{2} \frac{1}{3^{r+1}} + \frac{1\cdot 3}{2\cdot 4} \frac{1}{5^{r+1}} + \&c., \right)$$

$$\because \int_0^\infty (y^r e^{-ky} dy) = \frac{\lfloor r}{k^{r+1}}.$$

(ii) Similarly, $\int_0^{\frac{\pi}{2}} \log \sin \theta)^r \sec \theta \, d\theta = \int_0^\infty \frac{(-y)^r}{e^y - e^{q}} dy.$

$$= (-1)^r \lfloor r \left(\frac{1}{1^{r+1}} + \frac{1}{3^{r+1}} + \frac{1}{5^{r+1}} + \&c. \right)$$

$$= (-1)^r (2^{r+1}-1) \pi^{r+1} \frac{B_r + 1}{2(r+1)}.$$

(iii) $\int_0^{\frac{\pi}{2}} (\log \sin\theta)^r \frac{1+\tan\frac{1}{2}\theta}{1-\tan\frac{1}{2}\theta} d\theta$

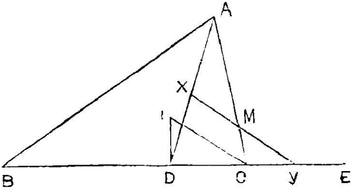
$$= \int_0^{\frac{\pi}{2}} (\log \sin\theta)^r \frac{\cos\theta}{1-\sin\theta} d\theta = \int_0^\infty \frac{(-y)^r}{ey-1} dy$$

$$= (-1)^r \lfloor r \left(\frac{1}{1^{r+1}} - \frac{1}{2^{r+1}} + \frac{1}{3^{r+1}} + \&c. \right)$$

$$= (-1)^r (2\pi)^{r+1} \frac{B_{r+1}}{2(r+1)}.$$

Question 337.

(S. NARAYANAN, B.A., L.T.) :—ABC is a triangle. D and E are the points of contact with BC of the incircle and the excircle corresponding to B, and I is the incentre. If X, Y be the midpoints of AD, CE respectively, shew that IC is parallel to XY.



Solution by M. Satyanarayana.

Let XY cut AC in M. Taking the $\triangle$ADC and the tranversal XMY

$$\frac{DX}{XA}\cdot\frac{AM}{MC}\cdot\frac{CY}{DY}\cdot=1$$

i.e., $\frac{AM}{MC}\cdot\frac{CY}{DY}\cdot=1$, since DX=XA.

$$\therefore\quad \frac{AM}{MC}=\frac{DY}{CY}$$

i.e., $\frac{AM+MC}{MC}=\frac{DY+CY}{CY}$

i.e., $\frac{AC}{MC}=\frac{DE}{CY}$

But AC=DE

$\therefore$ MC=CY.

Hence the result.

Analytical solution by G. Ramachandran.

Question 341.

(*Zero*) :—P,Q are inverse points with respect to a circle ; any st. line through P meets the diameter perpendicular to PQ in R. If QN be the perpendicular on this line, shew that RP.RN $=RA^2$ where A is the point of intersection of PQ and the circle.

Solution by G. Ramachandran, M. Satyanarayana and others.

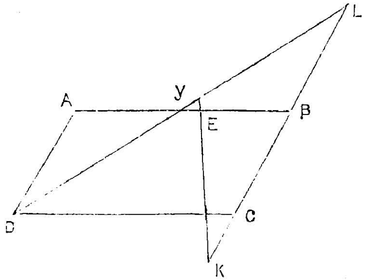
Let O be the centre of the circle.

Cince the $\triangle$s ROP, PNQ are similar

$$\begin{aligned} RP.PN &= OP.\ PQ \\ RP.RN &= OP.\ PQ+RP^2 \\ &= OP.\ OQ+RP^2-OP^2 \\ &= OA^2+OR^2 \\ &= RA^2 \end{aligned}$$

Question 342.

(A. A. Krishnaswami Aiyangar) :—ABCD is a parallelogram. A hyperbola is circumscribed to ABC so that one of its asymptotes passe through D. prove geometrically that the other asymptote touches a fixed hyperbola whose asymptotes are BA, BC.



Solution by R. Srinivasan, M.A.

Let the two asymptotes be DL and EK.

AB.BC = BC. AY + BC. BY and $\triangle$s AYD and BYL are similar.

$\therefore\ \frac{AY}{AD}=\frac{BY}{BL}\ \therefore$ AY. BL = AD. BY = BC. BY

$\therefore$ AB. BC = BC. AY + AY. BL = AY. CL

Since AB, BC, CA are chords of the hyperbola, CL = BK, AY = BE,

$\therefore$ B.K. BE is constant.

i. e., The area of $\triangle$ BKE is constant,

$\therefore$EK touches a hyperbola whose asymptotes are BA and BC.

QUESTIONS FOR SOLUTION.

344. (M. SATYANARAYANA) :—The incircle of a triangle ABC touches BC at D and X is the middle point of BC. Shew that AD is bisected by XI.

345. (M. SATYANARAYANA) :—P and Q are the isotomic conjugates of the ortho centre and the symmedian point of a triangle ABC, If AQ, BQ, CQ meet the opposite sides in D, E, F respectively, shew that PD, PE, PF are parallel to the symmedians of the triangle.

346. SRINIVASAN, M.A.) :—Shew that

$$\int_0^{\infty} \frac{1}{x} \tan^n u \sin 2n \cot^{-1}(m \tan x). \left\{ \sin \cot^{-1}(m \tan x) \right\}^{2n} dx$$

$$= \frac{\pi}{2} - \frac{1}{(1+m)^{2n}}$$

347. (PROF. SANJANA, M.A.) :—Denoting by S_r the sum of the series $\frac{1}{1^r}+\frac{1}{2^r}+\frac{1}{3^r}$...ad inf. ($r$ being a positive integer) prove that

$$\int_0^{\pi} \theta^k \log \operatorname{cosec} \theta.\, d\theta = \frac{\pi^{k+1}}{k+1} \log 2 + \pi^{k-1} s_3 - \frac{k(k-1)(k-2)}{2^4} \pi^{k-3} s_5$$

$$+ \frac{k(k-1)(k-2)(k-3)(k-4)}{2^6} \pi^{k-5} s_7 -$$

the series to stop short of the term in which the index of π or the co-efficient is zero.

348. (J. C. SWAMINARAYAN, M.A.) :—Two circles are described so as to touch a given circle of radius r, externally and internally with radii α and β respectively. If the given point is at a distance δ from the centre of the given circle, shew that

$$\frac{1}{\alpha} - \frac{1}{\beta} = \frac{4r}{\delta^2 - r^2}$$

349. (ZERO) :—A, B are two circles, such that the latter passes through the centre of the former. If C is any coaxal circle and D its inverse with respect to A, shew that the centres of C and D are inverse points with respect to B.

350. (K. J. SANJANA, M.A.) :—Sketch the curve given by $x = \phi \sin \alpha - \cos \phi$, $y = \cos \alpha + \sin \phi$ where ϕ is the parameter and find its inflexions.

Obtain the inflexions of

$$25x^3 = 25y^2 - 290xy - 504x^2 - 7000\, y.$$

351. (M. T. NARANIENGAR, M.A.) :—G is the centroid of a triangle ABC. AG, BG, CG are produced to meet the circumcircle in D,E,F; shew that G is a focus of the maximum inscribed ellipse of DEF.

352. (S. RAMANUJAN) :—Shew that

1\. (i) $$\frac{1}{1-}\,\frac{e^{-2\pi}}{1+}\,\frac{e^{-4\pi}}{1+}\,\frac{e^{-6\pi}}{1+\&c.}=\left(\sqrt{\frac{5+\sqrt{5}}{2}}-\frac{\sqrt{5}+1}{2}\right)\sqrt[5]{e^{2\pi}}.$$

(ii) $$\frac{1}{1-}\,\frac{e^{-\pi}}{1+}\,\frac{e^{-2\pi}}{1-}\,\frac{c^{-3\pi}}{1+\&c.}=\left(\sqrt{\frac{5-\sqrt{5}}{2}}-\frac{\sqrt{5}-1}{2}\right)\sqrt[5]{e^{\pi}}.$$

353. (S. RAMANUJAN) :—

2\. If n is any positive odd integer, shew that :—

$$\int_0^\infty \frac{\sin nx}{\cosh x+\cos x}\,\frac{dx}{x}=\frac{\pi}{4},$$ and hence prove that

$$\frac{\pi}{8}=\frac{1}{\cosh\frac{\pi}{2n}+\cos\frac{\pi}{2n}}-\frac{1}{3\left(\cosh\frac{3\pi}{2n}+\cos\frac{3\pi}{2n}\right)}+\frac{1}{5\left(\cosh\frac{5\pi}{2n}+\cos\frac{5\pi}{2u}\right)}-\&c.$$

for all odd values of n.

(S. NARAYANAN, B.A., L.T.) :—If P,Q,R be the points of trisection (next to the vertices) of the altitudes of a triangle ABC whose symmedian point is K, shew that the polar conic of K with respect to PQR is the locus of the centres of mean position of the feet of perpendiculars from any point on the circumcircle of ABC on its sides.

(N. SANKARA AIYAR) :—Solve completely.

$$x(1+x)^2\frac{d^2y}{dx^2}+x(1+x)\frac{dy}{dx}+y=x.$$

WANTED :

SOLUTIONS TO THE FOLLOWING .—

Questions—155, 231, 244, 249, 255, 256, 258, 262, 263, 266, 267, 269 270, 273, 274, 280, 284, 285, 289.